Paul Milgrom's work on Auctions and Information: A Retrospective

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Scope of this talk

- Theory of single-object auctions
  - Milgrom and Weber (1982) on symmetric auctions
  - Engelbrecht-Wiggans, Milgrom and Weber (1983) on informational asymmetries
- Plan
  - Brief account of preceding work
  - Contributions
  - Subsequent work on asymmetric auctions

In the beginning ...

- Vickrey (1961)
  - model of auctions as games of incomplete information
  - compare performance of different formats
    - expected revenue
    - efficiency

Vickrey (1961)

1. independent private values model
2. Dutch descending $\equiv$ first-price auction (FPA)
3. English ascending $\equiv$ second-price auction (SPA)
4. equilibrium of FPA (example)
5. revenue equivalence (example)
6. asymmetric first-price auctions (example)
7. multi-unit Vickrey auction

Revenue Equivalence Principle

- Fix an auction A such that only winner pays.
- Increasing equilibrium β^A
- $W^A(z)$ = expected price paid by *winner* who bids $\beta^A(z)$.
  - FPA
  $$W^{\text{FP}}(z) = \beta^{\text{FP}}(z)$$
  - SPA
  $$W^{\text{SP}}(z) = E[Y_1 \mid Y_1 < z]$$

Revenue Equivalence Principle

- Can show by direct computation that

$$\beta^{\mathrm{FP}}(z) = E[Y_1 \mid Y_1 < z]$$

  and so (Vickrey, 1961 & 1962):

$$W^{\mathrm{FP}}(z) = W^{\mathrm{SP}}(z)$$

- But, need to abstract away from specifics ...

Revenue Equivalence Principle

Theorem

If $W^A(0) = 0 = W^B(0)$, *then* $W^A(x) = W^B(x)$.

- **Proof:**
- Let $G(z) = \Pr[Y_1 < z]$.
- Bidder's problem

$$\max_z G(z)\,x - G(z)\,W^A(z)$$

- Optimal to set $z = x$, so

$$g(x)\,x = \left(G(x)\,W^A(x)\right)'$$

- So

$$\begin{aligned} W^A(x) &= \frac{1}{G(x)} \int_0^x y g(y)\,dy \\ &= E[Y_1 \mid Y_1 < z] \end{aligned}$$

IPV Model

Vickrey (1961)

Information Economics

Implementation Theory

Optimal Auction Design

Myerson (1981)

Riley and Samuelson (1981)

Common Value Model

- True value $V \sim H$
- Conditionally independent signals
  - $X_i \sim F(\cdot \mid V = v)$ i.i.d.
- Wilson (1967), Ortega-Reichert (1968) derived equilibrium in FPA (also examples with closed-form solutions)

MW's General Symmetric Model

- Interdependent values $v_i\left(x_1, x_2, \ldots, x_N, s\right)$
  - v_i symmetric in $\mathbf{x}_{-i}$
- Affiliated signals $f\left(x_1, x_2, \ldots, x_N, s\right)$
  - f symmetric in $\mathbf{x}$

MW's General Symmetric Model

- IPV model and CV model are special cases
- Affiliation assumption is key
    - inherited by order statistics
    - monotone functions

Main Results in MW

- Characterizing symmetric equilibria in FP, SP and English auctions
- $R^{\text{SP}} \geq R^{\text{FP}}$
    - $>$ with strict affiliation; private values OK
- $R^{\text{Eng}} \geq R^{\text{SP}}$
    - $>$ with strict affiliation, interdependence and $N > 2$
- $\widehat{R}^A \geq R^A$ Public information release (as a policy) increases revenue
- All standard auctions are *ex post* efficient
    - need single-crossing condition

IPV and MW

Symmetric IPV Model	MW Model
Dutch $\equiv$ FP	Dutch $\equiv$ FP
English $\equiv$ SP	$R^{\text{Eng}} \geq R^{\text{SP}}$
$R^{\text{SP}} = R^{\text{FP}}$	$R^{\text{SP}} \geq R^{\text{FP}}$
*	$\widehat{R}^{A} \geq R^{A}$

Equilibria of Standard Auctions

- Define $v(x, y) = E[V_1 \mid X_1 = x, Y_1 = y]$
- SPA:
$$\beta^{\text{SP}}(x) = v(x, x)$$
  - with private values $\beta^{\text{SP}}(x) = x$
- FPA:
$$\beta^{\text{FP}}(x) = \int_0^x v(y, y)\, dL(y \mid x)$$
where $L(\cdot \mid x)$ is determined by $G(\cdot \mid x)$
  - with private values $\beta^{\text{FP}}(x) = E[Y_1 \mid Y_1 < x]$

English Auction

- An *ex post* equilibrium is

$$
\begin{aligned}
\beta_N(x) &= v(x, x, ..., x) \\
\beta_{N-1}(x, p_N) &= v(x, x, ..., x, x_N) \\
&\vdots \\
\beta_k(x, \underbrace{p_{k+1}, ..., p_N}_{\text{Drop-out prices}}) &= v(x, x, ...x, \underbrace{x_{k+1}, ..., x_N}_{\text{Drop-out signals}})
\end{aligned}
$$

Given information inferred from drop-out prices, stay until price reaches value if all remaining bidders dropped out at this instant.

Revenue Ranking Results

- All the revenue ranking results, that is,

$$R^{\text{Eng}} \geq R^{\text{SP}} \geq R^{\text{FP}}$$

  can be deduced by direct computation from the equilibrium strategies.

- But, again helpful to abstract away from specifics ...

Linkage Principle

- Fix an auction A such that only winner pays.
- Increasing equilibrium β^A.
- $W^A(z,x)$ = expected price paid by *winner* who bids $\beta^A(z)$ when signal is x.
  - FPA
  $$W^{\mathrm{FP}}(z,x) = \beta^{\mathrm{FP}}(z)$$
  - SPA
  $$W^{\mathrm{SP}}(z,x) = E[\beta^{\mathrm{SP}}(Y_1) \mid X_1 = x, Y_1 < z]$$
- When is $W^A(x,x) \geq W^B(x,x)$?

Linkage Principle

Theorem

If (i) $W_2^A(x,x) \geq W_2^B(x,x)$*; and (ii)* $W^A(0,0) = 0 = W^B(0,0)$*, then*

$$W^A(x,x) \geq W^B(x,x)$$

- **Proof:**
- Let $G(z \mid x) = \Pr[Y_1 < z \mid X_1 = x]$.
- Bidder's problem in auction A

$$\max_z \int_0^z v(x,y)\, g(y \mid x)\ dy - G(z \mid x)\, W^A(z,x)$$

- Optimal to set $z = x$, so

$$W_1^A(x,x) = \frac{g(x \mid x)}{G(x \mid x)} v(x,x) - \frac{g(x \mid x)}{G(x \mid x)} W^A(x,x)$$

Linkage Principle

Similarly, in auction B :

$$W_1^B(x,x) = \frac{g(x \mid x)}{G(x \mid x)} v(x,x) - \frac{g(x \mid x)}{G(x \mid x)} W^B(x,x)$$

If we write

$$\Delta(x) = W^A(x,x) - W^B(x,x)$$

then

$$\Delta'(x) = -\frac{g(x \mid x)}{G(x \mid x)} \Delta(x) + \left[W_2^A(x,x) - W_2^B(x,x)\right]$$

Since $\Delta(0) = 0$ and $\Delta(x) < 0$ implies $\Delta'(x) > 0$, we have $\Delta(x) \geq 0$.

Public Information Release

- $W^{\text{FP}}(z,x) = \beta^{\text{FP}}(z)$
  - so $W_2^{\text{FP}}(x,x) = 0$
- $\widehat{W}^{\text{FP}}(z,x) = E\left[\beta^{\text{FP}}(z,S) \mid X_1 = x\right]$
  - so by affiliation $\widehat{W}_2^{\text{FP}}(x,x) \geq 0$
- Linkage principle now implies that $\widehat{R}^{\text{FP}} \geq R^{\text{FP}}$
- Similar argument for $R^{\text{Eng}} \geq R^{\text{SP}}$

Theory and Policy

- Affiliation is key for existence of monotone pure strategy equilibria in FPA in asymmetric situations
  - Athey (2001)
  - Reny & Zamir (2004)
  - de Castro (2007) ("just right")
- Affiliation + linkage principle $\rightarrow$ advantages of open auctions
  - market design in other settings

Empirical Work and Experiments

- Hendricks, Pinkse and Porter (2003) use ex post value data to show that bidding in (symmetric) off-shore oil auctions is consistent with equilibrium of MW model.
- Kagel and Levin's (2002) extensive work on experiments concerning MW model.

An Impossible Ideal

- Beautiful deep theory
- Clean results
- Strong policy recommendations (open auctions, transparency)
- Empirical support

Generalizations?

- Can the linkage principle be generalized to accommodate
  - asymmetries among bidders?
  - symmetric multi-unit auctions?
- The two are closely related: even symmetric multi-unit auctions lead to asymmetries
  - my bid for first unit may compete with your bid for second unit

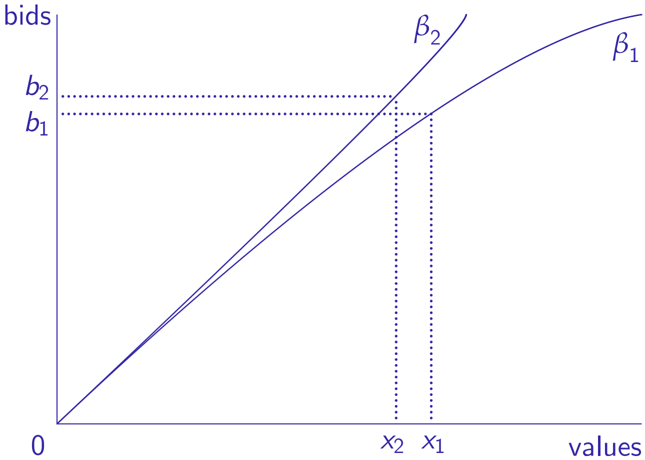
Asymmetries and Revenue Rankings

- Even with asymmetric *independent* private values ($F_1 \neq F_2$) we know that

  $$R^{\text{FP}} \gtrless R^{\text{SP}}$$

  Vickrey (1961)

- Ranking depends on distributions
  - $R^{\text{FP}} \gtrless R^{\text{SP}}$ even if F_1, F_2 are
    - stochastically ranked
    - regular
    - (truncated) Normals
  - Maskin and Riley (2000) classification.
- Also, FP is inefficient.


bids
β_2
β_1
b_2
b_1
0
x_2
x_1
values

Resale

- Inefficiency leads to possibility of resale.
  - a simple model:
- Stage 1: First-price auction
  - Price (winning bid) is announced
- Stage 2: Winner (new owner) makes a take-it-or-leave-it offer to other buyer
  - Note resale takes place under incomplete information, so still inefficient

Resale

Theorem

Suppose $N = 2$ and F_1, F_2 regular. Then with resale

$$\overline{R}^{FP} \geq \overline{R}^{SP}$$

- Hafalir and Krishna (2008)
- Extensions to $N > 2$?

Public Information with Asymmetries: Example

Suppose X_1, X_2, S uniform i.i.d. and

$$\begin{array}{rcll} v_1(x_1, x_2, s) & = & x_1 + \frac{1}{2}(x_2 + s) & \text{interdependent} \\ v_2(x_1, x_2, s) & = & x_2 & \text{private} \end{array}$$

- With no information release by seller, equilibrium in SPA

$$\beta_1(x_1) = 2x_1 + E[S] \text{ and } \beta_2(x_2) = x_2$$

- With information release,

$$\widehat{\beta}_1(x_1, s) = 2x_1 + s \text{ and } \widehat{\beta}_2(x_2) = x_2$$

Example (contd.)

- Given x_1 and x_2, the (expected) prices are

$$\begin{aligned} P &= \min\left\{2x_1 + E\left[S\right], x_2\right\} \\ \widehat{P} &= E\left[\min\left\{2x_1 + S, x_2\right\}\right] \end{aligned}$$

- But "min" is a concave function and so $\widehat{P} < P$.
- In this example, release of information $S = s$ *decreases* revenue in a SPA:

$$\widehat{R}^{\mathrm{SP}} < R^{\mathrm{SP}}$$

- Similar failure of linkage principle in multi-unit auctions (Perry and Reny, 1999)

Asymmetries and Revenue Rankings: Example

Suppose

$$\begin{aligned} v_1(x_1, x_2, x_3) &= \tfrac{1}{2}x_1 + \tfrac{1}{2}x_2 \quad \text{common} \\ v_2(x_1, x_2, x_3) &= \tfrac{1}{2}x_1 + \tfrac{1}{2}x_2 \quad \text{common} \\ v_3(x_1, x_2, x_3) &= x_3 \quad \text{private} \end{aligned}$$

X_1, X_2, and X_3 are i.i.d. uniform on $[0, 1]$.

- In this example

$$R^{\text{Eng}} < R^{\text{SP}}$$

- Revenue rankings do not generalize to asymmetric situations.

From Revenue to Efficiency

- MW paper derives very general and powerful results on revenue comparisons in *single*-object *symmetric* settings.
- As the examples show, general revenue ranking results are unlikely to hold in more general situations
  - for instance, question regarding treasury bill auctions (discriminatory vs. uniform-price) remains open
- Auction theory has turned to the question of efficiency
  - much of this work is about the efficient allocation of *multiple* objects in a private value setting (Larry Ausubel's talk)
  - but question of allocating *single* objects in *asymmetric* settings with *interdependent* values remains

Efficient Allocations

- Suppose we have N buyers with values $v_i(x_1, x_2, ..., x_N)$
- *Ex post* efficiency means that if i gets object then $v_i(x_1, x_2, ..., x_N) \geq v_j(x_1, x_2, ..., x_N)$ for $j \neq i$.
- Maskin (1992) suggested that English auctions may allocate efficiently in asymmetric settings
    - Proof for $N = 2$ (under single-crossing)

Efficiency under Asymmetries

- Step 1: solve for inverse bidding strategies ϕ_1 and ϕ_2 such that

$$\begin{aligned} v_1\left(\phi_1\left(p\right), \phi_2\left(p\right)\right) &= p \\ v_2\left(\phi_1\left(p\right), \phi_2\left(p\right)\right) &= p \end{aligned}$$

  - Single-crossing guarantees monotone solution
- Step 2: If $p_1 > p_2$ (1 wins), then we have
  - $x_1 = \phi_1\left(p_1\right) > \phi_1\left(p_2\right)$ (mono.)
  - $x_2 = \phi_2\left(p_2\right)$
  - So

$$\begin{aligned} v_1\left(x_1, x_2\right) &= v_1\left(\phi_1\left(p_1\right), \phi_2\left(p_2\right)\right) \\ &> v_1\left(\phi_1\left(p_2\right), \phi_2\left(p_2\right)\right) \\ &= p_2 \end{aligned}$$

Efficiency under Asymmetries

- We have argued that there is an *ex post* equilibrium (distribution-free)
- Is this *ex post* efficient?
    - Yes:

$$\begin{aligned}
v_1(\phi_1(p_2), \phi_2(p_2)) &= v_2(\phi_1(p_2), \phi_2(p_2)) = p_2 \\
v_1(\phi_1(p_2), x_2) &= v_2(\phi_1(p_2), x_2) \\
v_1(\phi_1(p_1), x_2) &> v_2(\phi_1(p_1), x_2) \text{ (SC)} \\
v_1(x_1, x_2) &> v_2(x_1, x_2)
\end{aligned}$$

English Auctions

- Maskin's two-person result does not extend without strengthening SC conditions (how my signal affects aggregate value).

Theorem

Suppose single crossing in the "aggregate" is satisfied. Then the English auction has an efficient ex post equilibrium.

- Krishna (2002) (also, Wilson's (1998) log-normal model)
- Dubra, Echenique and Manelli (2009) have recently provided weaker sufficient (and almost necessary) conditions.
- The constructions generalize the *ex post* equilibrium construction in MW

English Auctions

- Milgrom and Weber advocated English auctions on *revenue* grounds (Linkage Principle)
  - revenue results do not extend to asymmetric situations, but ...
- It turns out that even in asymmetric situations open auctions have remarkable *efficiency* properties!

Open Auctions

- The general message that open auctions are advantageous is powerful and still resonates in more general and realistic settings.
- Bravo English auctions!
- Bravo Paul Milgrom!

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