2017 Microeconomics Prelim

This exam is divided into three question: one each for 410-1, 410-2, and 410-3.

Please note: EACH QUESTION HAS THE SAME WEIGHT IN THE EXAM, REGARDLESS OF THE MAXIMUM POINTS POSSIBLE IN THAT QUESTION. Each question is divided into parts and sub-parts. Partial credit is awarded as indicated. You should attempt to solve each question.

410-1 Question

Note: You should concisely but clearly justify all your answers.

1. **(10 points)** Assume F_1 and F_2 are CDFs with common support contained in $[0,1]$.

   In the first part below assume both CDFs have finite support. Furthermore, for $i=1,2$, denote the smallest point in the common supports by $\underline{x}$ and the largest by $\bar{x}$.

   In the second and third parts below assume they are both atomless distributions with densities f_1 and f_2 and common support equal to $[0,1]$.

   (a) **(2 points)** Assume F_1 dominates F_2 according to FOSD. Is it true that for *any* such F_1 and F_2 there is an $x \in (\underline{x}, \bar{x})$ (i.e., strictly greater than the lowest point in the common supports and strictly below the highest one) such that F_1 conditional on $[0,x]$ FOSD F_2 conditional on $[0,x]$ *and* F_1 conditional on $[0,x]$ is not equal to F_2 conditional on $[0,x]$? If yes, specify how you find the x, if not argue why there is a counter-example.

   (b) **(4 points)** Assume that F_2 is a mean-preserving increase in risk of F_1. Is it true that for any such F_1 and F_2 there is an $x \in (0,1)$ such that F_1 conditional on $[0,x]$ FOSD F_2 conditional on $[0,x]$? If yes, specify how you find the x, if not argue why there is a counter-example.

   (c) **(4 points)** Assume that F_1 MLR dominates F_2. Is it true that for any such F_1 and F_2 there is an $x \in (0,1)$ such that F_1 conditional on $[0,x]$ FOSD F_2 conditional on $[0,x]$? If yes, specify how you find the x, if not argue why there is a counter-example.

2. **(10 points)** Consider a production problem, where a price-taking firm with production function $f : R^n_+ \to R_+$ is maximizing profits. The price of output is $p > 0$, however the cost of inputs is not necessarily linear; specifically, the price of input vector z is $w(z)$, where $w : R^n_+ \to R_+$ is a strictly increasing and continuous function. For a given production function f define the following four functions.

   The cost function is $C(q, w(\cdot)) = \min w(z)$ s.t. $f(z) \geq q$

   The input demand function $z^*(q, w(\cdot))$ is a solution to the cost-function problem above.

   The profit function is $\pi(p, w(\cdot)) = \max_z pf(z) - w(z)$

   The supply function is given by substituting a solution to the preceding problem, denoted $\hat{z}(p, w(\cdot))$, into the production function: $q^*(p, w(\cdot)) = f(\hat{z}(p, w(\cdot)))$.

We are interested in finding what assumptions on $w(\cdot)$ and $f(\cdot)$, in addition to being strictly increasing and continuous, are necessary and sufficient so that for any f and any strictly positive p the following standard properties and results hold. Hereafter, the convex combination of two input cost functions w_1 and w_2 is defined by $[aw_1 + (1-a)w_2](z) = aw_1(z) + (1-a)w_2(z))$. Finally given functions $g : R^n \to R$ and $h : R \to R$ we will say that $h(g(\cdot))$ is homogenous of degree t in g if $h(kg(\cdot)) = k^t h(g(\cdot))$ for any $k > 0$.

(a) **(4 points)** Homogeneity:

   i. **(2 points)** Consider a standard (as studied in class) production problem, i.e., as above where w is linear. Specify which of the four defined functions are homogenous in what degree and in what variables and under what additional assumptions on $f(\cdot)$.

   ii. **(2 points)** Specify what additional assumption, if any, on $w(\cdot)$ is necessary and sufficient for these results to continue to hold in the current (non-linear input prices) context.

(b) **(6 points)** Single-valuedness:

   i. **(3 points)** Consider a standard (as studied in class) production problem, i.e., as above where w is linear. Specify which of the four defined functions are single-valued and if needed under what additional assumption(s) on $f(\cdot)$.

   ii. **(3 points)** Specify what pair(s) of independent assumptions on $f(\cdot)$ and $w(\cdot)$ are necessary and sufficient for this in the current (non-linear input prices) context?
   (Here by independent assumptions I mean that the assumption on $f(\cdot)$ cannot depend on the specification of $w(\cdot)$ and conversely, e.g. you cannot say $f(\cdot)$ has to be a polynomial of $w(\cdot)$). By necessary and sufficient I mean that if either $f(\cdot)$ or $w(\cdot)$ violates the assumption you specify then there is an example where the other does not violate the assumption and the conclusion fails.)

410-2 Question

Consider an economy with spot trading under asymmetric information. There are $I \geq 2$ consumers and two commodities, "food" x and "money" m. The state space is $S = \prod_{i \in I} S_i$, where $S_i = [0, 1]$; assume a uniform distribution over S. Consumer i observes s_i, but not the other coordinates of S. [In other words, i's information partition is $\mathcal{I}_i = \{\{s \in S : s_i = \bar{s}_i\} \ : \ \bar{s}_i \in S_i\}$.]

Every consumer's state-dependent Bernoulli utility $u_i : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is defined by

$$u_i(x_i, m_i) = \left(\alpha_i s_i + \sum_{j \neq i} s_j \right) \ln x_i + m_i,$$

where $\alpha_i > 0$ for all i; unless otherwise specified, these values can be different for different agents. Every consumer has an endowment of 1 unit of food and zero units of money in every state.

As usual, normalize the price of money in every state to 1; a (REE or pooled-information equilibrium) price function is then a map $p : S \to \mathbb{R}$. [NOTE: although the state space is uncountable, and you will need to condition on lower-dimensional subsets of S, conditional expectation works in the "obvious" way. Everything can be made rigorous, but don't worry about technicalities.]

(a) (3 points). Show that there is no REE in which the price function is constant (i.e., such that, for some $p^* \in \mathbb{R}$, $p(s) = p^*$ for all $s \in S$.)

(b) (3 points). Compute the unique pooled-information equilibrium price function. Is it fully revealing?

(c) (3 points). Show that the pooled-information equilibrium price function in (b) is also a REE price function if either (1) $I = 2$, or (2) there is $\bar{\alpha} > 0$ such that $\alpha_i = \bar{\alpha}$ for all i.

The next three parts ask you to show that, beyond the cases considered in (c), the pooled-information equilibrium price function you found in (b) is *not* also a REE price function. We consider the case of $I = 3$ and assume that $\alpha_1 < \alpha_2 < \alpha_3$. **NOTE:** if you get stuck on one part, assume the claim is true and **move on**. Remember, partial credit will be given!

(d) (2 points). Suppose, by contradiction that $p(\cdot)$ in part (b) is a REE price function. [The contradiction will be established in part (f).] Show that then, for any consumer i and state s^*, if the conditional expectation of $\sum_{j \neq i} s_j$ given that $p(s) = p(s^*)$ and $s_i = s_i^*$ is strictly greater than $\sum_{j \neq i} s_j^*$, then, in state s^*, i's demand for x is strictly greater in the REE than in the pooled-information equilibrium. [HINT: consider what is uncertain in the consumer's state-dependent Bernoulli utility function, and what she conditions on in a REE and in a pooled-information equilibrium.]

(e) (3 points). Show that, for any state s^*, the set of states s such that $p(s) = p(s^*)$ and $s_i = s_i^*$ is the Cartesian product of $\{s_i^*\}$ with a segment in the unit square $\prod_{j \neq i} S_j = [0,1]^2$. That is, show that, for every s^*, and every permutation (i, j, k) of $(1, 2, 3)$, there exist $a, b \in \mathbb{R}$ such that

$$\{s \in S \,:\, p(s) = p(s^*),\, s_i = s_i^*\} = \{s_i^*\} \times \{(s_j, s_k) \in S_j \times S_k = [0,1]^2 \,:\, s_j = a + b s_k\}.$$

Argue that, therefore, the expectation of $\sum_{j \neq i} s_j$ conditional upon $p(s) = p(s^*)$ and $s_i = s_i^*$ is the value of $\sum_{j \neq i} s_j$ at the midpont of this segment.

(f) (6 points). Consider state $s^* = (0, \frac{1}{2}, 1)$. Use (e) to show that, for every $i = 1, 2, 3$, the conditional expectation of $\sum_{j \neq i} s_j$ given that $p(s) = p(s^*)$ and $s_i = s_i^*$ is strictly greater than $\sum_{j \neq i} s_j^*$. Argue that, therefore, by (d), $p(\cdot)$ cannot be a REE price function. **NOTES**: (1) You need to consider each i separately—so there are three cases. You get **2 points** for each case, so it's better to solve one case fully then three cases badly; $i = 2$ is the easiest. (2) In principle, there may be a clean general proof that applies to all i, but I do not know of one.

COMMENT (not necessary to answer this question): in case you worry about these things, the inequalities you will find in (f) are strict. Since all quantities of interest are continuous in s, the same inequalities hold in a neighborhood of $(0, \frac{1}{2}, 1)$. So the failure of $p(\cdot)$ to be a REE function is not just a knife-edge case. Indeed similar failures arise in other regions of the state space; the calculations in this particular region just happen to be a little bit easier to carry out.

410-3 Question

Suppose the Government wants to allocate spectrum to the private sector. To make things simple, assume the Government possesses a single, indivisible, license. There are N potential buyers, each with type θ_i drawn from an absolute continuous distribution F_i, with density f_i strictly positive over $[\underline{\theta}_i, \bar{\theta}_i]$. The values $\theta \equiv (\theta_i)_{i=1}^{i=N}$ are drawn independently. Formally, let $X = \{x \equiv (x_i)_{i=1}^{i=N} \in \{0,1\}^N : \sum_{i=1}^{i=N} x_i \leq 1\}$ denote the set of possible allocations (Note that $x = (0, ..., 0)$ corresponds to the decision to not allocate the good to any of the buyers). Each buyer's gross payoff is given by $v_i(x, \theta_i)$, whereas the Government's gross payoff is given by the function $v_g(x, \theta_g)$. Note that the above functions capture arbitrary externalities from the assignment of the license. Importantly, throughout the entire question, the Government is *not* allowed to exchange money with the private sector.

(a) **(8 pts)** Construct a Bayesian incentive-compatible (BIC) mechanism that, in each state, implements the assignment that maximizes the sum of the buyers' and the Government's gross payoffs, and (B) is budget balanced (among the agents). [Hint: in this problem, the Government has (known) preferences over the allocation of the good, but is not allowed to exchange money with the other players. Yet, a simple adaptation of a mechanism we studied in class does the job].

(b) **(8 pts)** Suppose now that the buyers' payoffs take the form

$$v_i(x, \theta_i) = \begin{cases} \theta_i & \text{if } x \text{ s.t. } x_i = 1 \\ 0 & \text{if } x \text{ s.t. } x_i = 0 \end{cases}$$

for all $i = 1, ..., N$. Likewise, the Government's payoff takes the form

$$v_g(x, \theta_g) = \begin{cases} \theta_g & \text{if } x \text{ s.t. } x_i = 0 \text{ all } i = 1, ..., N \\ 0 & \text{otherwise.} \end{cases}$$

Further assume that, for all $i = 1, ..., N$, $F_i = F$ (i.e., the buyers' types are iid random draws) and $\underline{\theta} > 0$. Assume each buyer's outside option is equal to zero. Show that, in this case, the mechanism constructed in part (a) is also interim individually rational (IIR).

(c) **(4 pts)** Show whether or not, under the specification in part (b), the mechanism constructed in part (a) is also dominant strategy incentive compatible (DSIC).